Thermodynamic Bethe Ansatz for Fishnet CFT

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2020/08/26 IGST2020 ICTP-SAIFR

Phys. Rev. Lett. **125**, 091601 ArXiv: 1911.10213
with Benjamin Basso, Gwenaël Ferrando, Vladimir Kazakov

Fishnet Theory in 4D

For review, see [Kazakov '18]

❖ **Chiral, non-unitary theory in 4D** [Gürdoğan, Kazakov '15]

$$\mathcal{L} = N_c\, \mathtt{Tr} \left(\partial_\mu \phi_1 \partial^\mu \phi_1^\dagger + \partial_\mu \phi_2 \partial^\mu \phi_2^\dagger + (4\pi\xi)^2 \phi_1 \phi_2 \phi_1^\dagger \phi_2^\dagger \right)$$

$\phi_j \in \mathrm{Mat}(N_c)$

❖ **Conformal symmetry**

- even with quantum corrections [Fokken,Sieg,Wilhelm '14] [Sieg-Wilhelm '16] [Grabner, Gromov, Kazakov, Korchemsky '17]

❖ **Planar integrability**

- Conformal spin chain [Chicherin, Derkachov, Isaev '12]
- Partial resummation of Feynman diagrams

❖ **New kind of 4d integrable CFT**

❖ **Feynman diagrams are much simpler!**

Generalisation to Any Dimension

- **Two ways of generalising:**

Higher dimensional (D>2)

(1) Using other integrable ~~theories~~: ABJM in 3D
theory

[Caetano, Gürdoğan, Kazakov '16]

$$\mathcal{L} = N_c \, \mathrm{Tr} \left[-\partial_\mu Y_1^\dagger \partial^\mu Y^1 - \partial_\mu Y_2^\dagger \partial^\mu Y^2 - \partial_\mu Y_4^\dagger \partial^\mu Y^4 \right] + \mathcal{L}_{\mathrm{int}}$$

$$\mathcal{L}_{\mathrm{int}} = (4\pi)^2 \, g \, \mathrm{Tr}(Y^1 Y_4^\dagger Y^2 Y_1^\dagger Y^4 Y_2^\dagger)$$ $\lambda\phi^6$ -type interaction

6D analog still missing! $\lambda\phi^3$-type interaction

Generalisation to Any Dimension

- **Two ways of generalising:**

  (2) Integrable $SO(1, D+1)$ conformal spin chain:

  [Chicherin, Derkachov, Isaev '12]

  - **Lagrangian description** [Kazakov, Olivucci '18]

$$\mathcal{L} = N_c \mathtt{Tr}\left[\phi_1^\dagger \left(-\partial_\mu \partial^\mu\right)^{\tilde{\delta}} \phi_1 + \phi_2^\dagger \left(-\partial_\mu \partial^\mu\right)^{\delta} \phi_2 + (4\pi)^{\frac{D}{2}} \xi^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right]$$

$$\tilde{\delta} = D/2 - \delta, (0 < \delta < D/2)$$

  - **Conformal & Integrable**
  - **No "parent" theory! No crossing!**
  - **Need to find a way to develop the integrability machinery**

Scattering Data of TBA

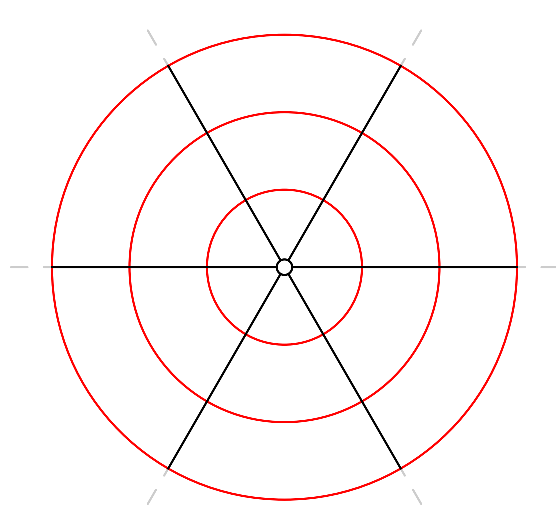
- **Integrability determines spectrum:** Theormodynamic Bethe Ansatz

- **TBA requires scattering data:** dispersion & S-matrix

- **Observables:** scaling dimension of

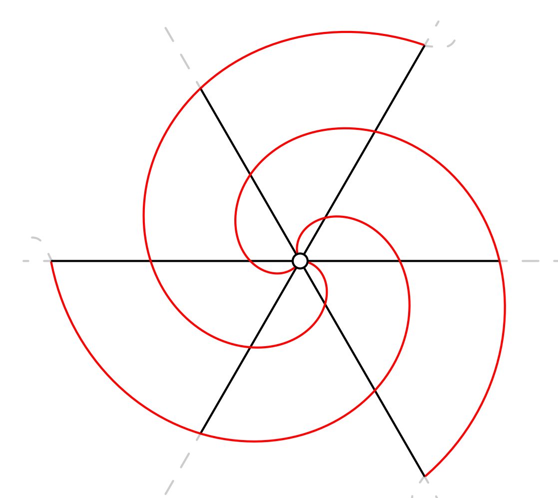
$$\mathcal{O}_{J,M}(x) = \mathtt{Tr}(\phi_1^M \phi_2^J) + \dots .$$

- **Typical Feynman diagrams:** 2pt function, “wheel” & “spiral”



$(M = 0)$

“Vacuum”



$(M \neq 0)$

“Excited states”

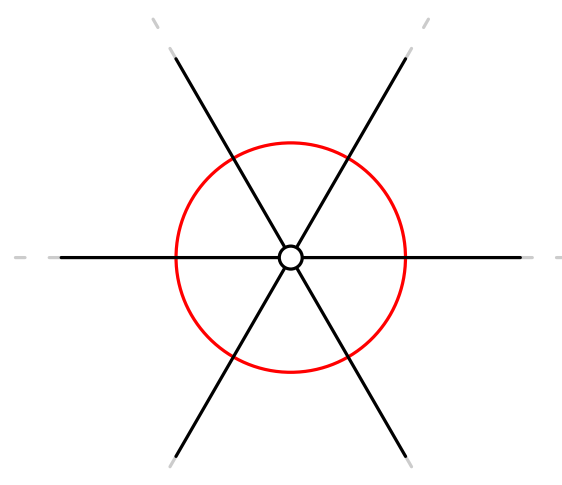
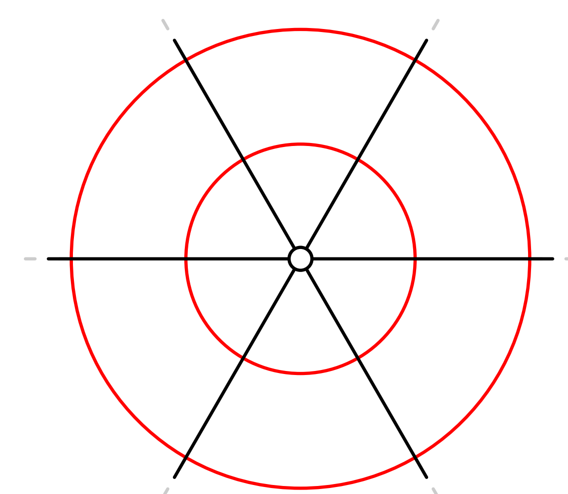
Wheel Diagrams in Perturbation Theory

❖ **Let's study the vacuum state first!**

❖ **Probe:** scaling dimension of $\mathcal{O}_{M=0,J} = \mathtt{Tr}\,\phi_2^J \quad (J > 2)$

$\Delta = J + \gamma$ Protected until the wrapping order

❖ **Renormalisation:** summing wheel diagrams

$Z = 1 +$  $+$  $+ \cdots$

wave-function renormalisation

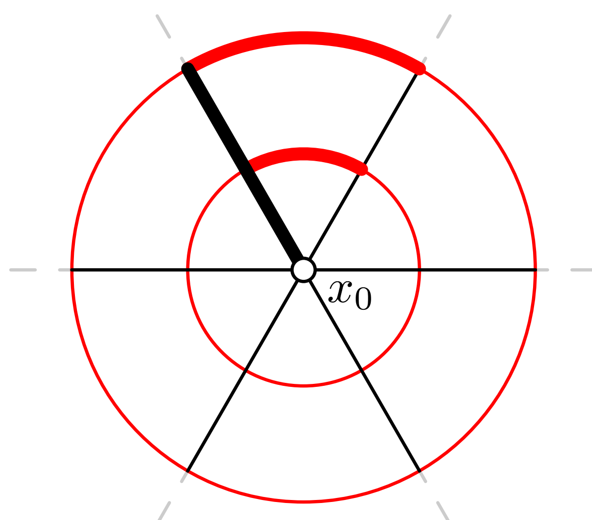
$O(\xi^{2J})$ $O(\xi^{4J})$

$$\log Z \sim -\gamma \times \log \Lambda_{UV}$$

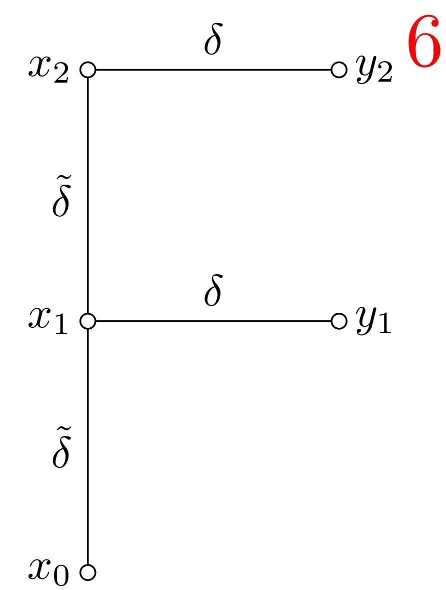
The Graph Building Operator

❖ **Feynman diagrams as repeated action of some integral operators**

2D Case, see [Derkachov, Kazakov, Olivucci '19]

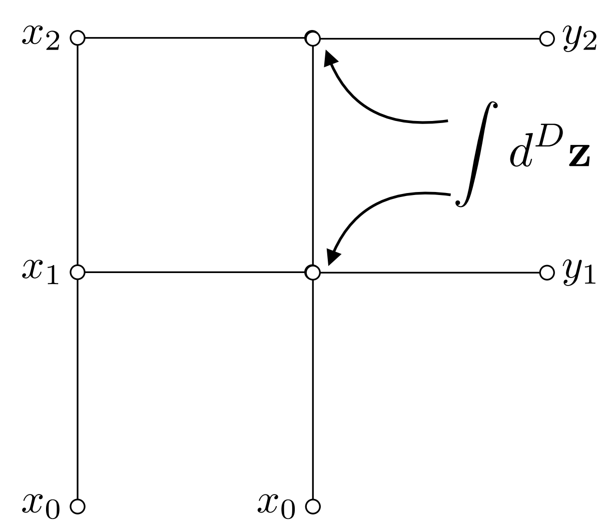


$$= \operatorname{Tr}\Gamma^{6}_{N=2} =$$



❖ **Repeated action by integrating:** for example,

$$\Gamma^2(\mathbf{x}|\mathbf{y}) = \int d^D\mathbf{z}\,\Gamma(\mathbf{x}|\mathbf{z})\Gamma(\mathbf{z}|\mathbf{y})$$

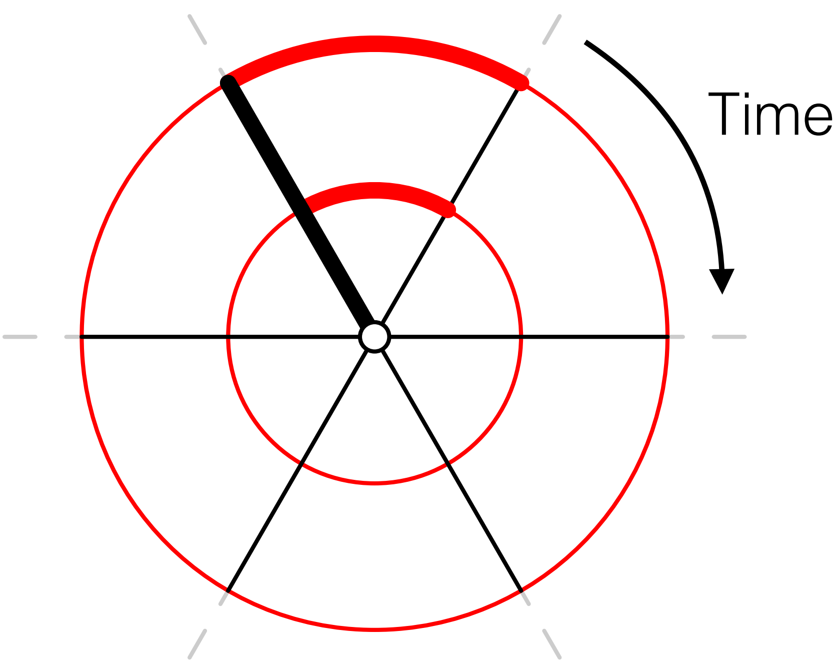


Magnon Picture in the Mirror Channel

❖ **Dictionary from field theory to integrability**

❖ **Magnon:** integrability description

- **Magnon**: ϕ_1 propagators
- Effective 1D problem: $\underset{r=e^{\sigma}}{\mathbb{R}^D \cong \mathbb{R}_+} \times \underset{O(D)}{S^{D-1}}$
- "**Space**": radial direction
- "**Time**": angular direction
- Time evolution: repeated action of Γ_N



Scattering Data from Graph Building Operators

❖ **Magnon scattering data is related to eigensystem of** Γ_N

- **Energy**: the weight conjugate to time = Eigenvalue of $\Gamma_{N=1}$
- N-body **wave function**: eigenfunction of Γ_N
- **S-matrix**: "relative phase" of two magnon scattering when $x_{12}^2 \to \infty$

$$\Psi \simeq e^{ip_1 \log|x_1| + ip_2 \log|x_2|} C\left(\frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}\right) + e^{ip_2 \log|x_1| + ip_1 \log|x_2|} \left[\mathbb{S}_{l_1,l_2}(u_1,u_2) C\right]\left(\frac{x_2}{|x_2|}, \frac{x_1}{|x_1|}\right)$$

O(D) d.o.f

Assuming Integrability, those data fully determine the spectrum!

Magnon Dispersion Relation

- **The eigensystem for $\Gamma_{N=1}$ is simple to obtain**

  - Eigenvector:

$$\Psi_1 = \frac{1}{x^{2\tilde{\beta}}} C\left(\frac{x}{|x|}\right), \quad C(y) = C^{\mu_1 \dots \mu_l} y_{\mu_1} \dots y_{\mu_l}$$

Symmetric Traceless Tensor

$\tilde{\beta} = (D-\delta)/2 - iu, \ u \in \mathbb{R}$

  - Eigenvalue:

$$\Gamma_{N=1}\Psi_1 = \lambda_l(u)\Psi_1, \quad \lambda_l(u) = \frac{\Gamma(\tilde{\delta})}{\Gamma(\delta)} \frac{\Gamma\left(\frac{\delta}{2} + \frac{l}{2} + iu\right)\Gamma\left(\frac{\delta}{2} + \frac{l}{2} - iu\right)}{\Gamma\left(\frac{D-\delta}{2} + \frac{l}{2} + iu\right)\Gamma\left(\frac{D-\delta}{2} + \frac{l}{2} - iu\right)}$$

- **We can read out magnon dispersion !**

$$p_l(u) = 2u, \quad \varepsilon_l = -\log \lambda_l$$

Variable conjugate to radial distance

Weight of time evolution $e^{-\varepsilon_l} = \lambda_l$

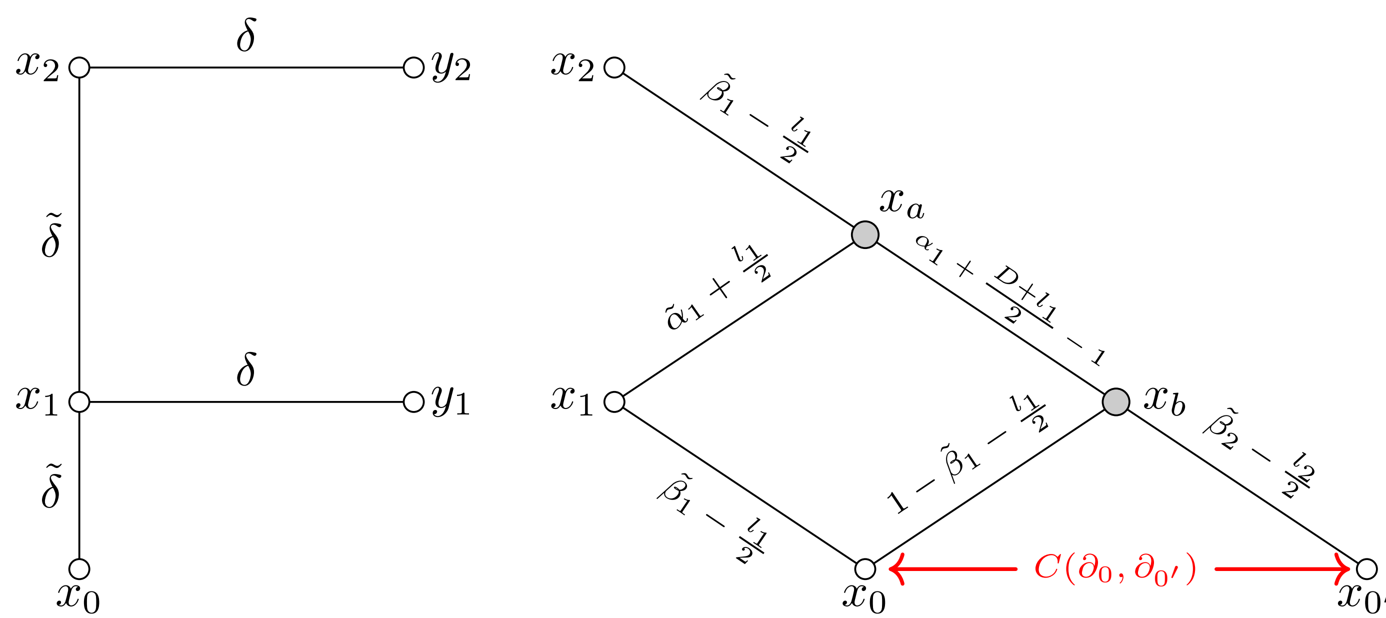
Magnon S-Matrix

❖ **Expectation:** (S-matrix depends on rapidity and integer spin)

(1) Matrix part is fully determined by Yang-Baxter: O(D) symmetry

(2) Eigenvalue factorise: total energy is the sum of individual energies

❖ **Eigenvector for N = 2 case:**



$$\alpha_j = D/2 - \tilde{\alpha}_j = \delta/2 - iu_j, \; \beta_j = D/2 - \tilde{\beta}_j = \delta/2 + iu_j$$

$$C(y_1, y_2) = C^{\mu_1 \ldots \mu_{l_1} \nu_1 \ldots \nu_{l_2}} y_{1\mu_1} \ldots y_{1\mu_{l_1}} y_{2\nu_1} \ldots y_{2\nu_{l_2}}$$

Symmetric traceless tenor, separately, in the first l_1 indices and in the last l_2 ones

Magnon S-Matrix: Results

- **When two magnons are far apart, $x_{12}^2 \to \infty$**

$$\Psi \simeq (x_1^2)^{iu_1}(x_2^2)^{iu_2} C\left(\frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}\right) + (x_2^2)^{iu_1}(x_1^2)^{iu_2}\left[\mathbb{S}_{l_1,l_2}(u_1,u_2)C\right]\left(\frac{x_2}{|x_2|}, \frac{x_1}{|x_1|}\right)$$

$$\boxed{\mathbb{S}_{l,l'}(u,v) = \frac{f_l(u)}{f_{l'}(v)}\mathcal{S}_{l,l'}(u-v)\mathbb{R}_{l,l'}(u-v)}$$

R-matrix satisfies Yang-Baxter relation

$$\mathcal{S}_{l,l'}(u) = \frac{\Gamma(1+\frac{l+l'}{2}-iu)}{\Gamma(1+\frac{l+l'}{2}+iu)}\frac{\Gamma(\frac{D}{2}+\frac{l+l'}{2}+iu)}{\Gamma(\frac{D}{2}+\frac{l+l'}{2}-iu)} \times \frac{\Gamma(\frac{|l-l'|}{2}+iu)}{\Gamma(\frac{|l-l'|}{2}-iu)}\frac{\Gamma(1+\frac{|l-l'|}{2}+iu)}{\Gamma(1+\frac{|l-l'|}{2}-iu)}$$

$$f_l(u) = \frac{\Gamma\left(\frac{\delta}{2}+\frac{l}{2}-iu\right)\Gamma\left(\frac{D-\delta}{2}+\frac{l}{2}-iu\right)}{\Gamma\left(\frac{\delta}{2}+\frac{l}{2}+iu\right)\Gamma\left(\frac{D-\delta}{2}+\frac{l}{2}+iu\right)}$$

- **We can partly verify this conjecture. Including reproducing**

$$\mathbb{R}_{1,1}(u) = \frac{u\mathbb{I}}{u+i} + \frac{i\mathbb{P}}{u+i} - \frac{iu\mathbb{K}}{(u+i)(u+i\frac{D-2}{2})}$$

[Zamolodchikov^2 '79]

Identity Permutation Trace

TBA System

❖ **TBA equations are fully determined by the scattering data**

- Scaling dimension = free energy of magnons
- Length = Inverse temperature $T = \frac{1}{J}$
- Coupling constant = Chemical potential $\mu = \log \xi^2$

$$\Delta = J\tilde{\delta} - \sum_{l \geq 0} \int_{\mathbb{R}} \frac{du}{2\pi} p_l'(u) \log\left(1 + \mathrm{Y}_{1,l}(u)\right)$$

Y function: distribution of energy per magnon

Rotational d.o.f: spherical harmonics

TBA for O(D+2) (D even)

❖ **TBA equations:** infinite system of non-linear equations determine Y

- Massive Y-functions: $(a=1, l \geq 0)$

Convolution

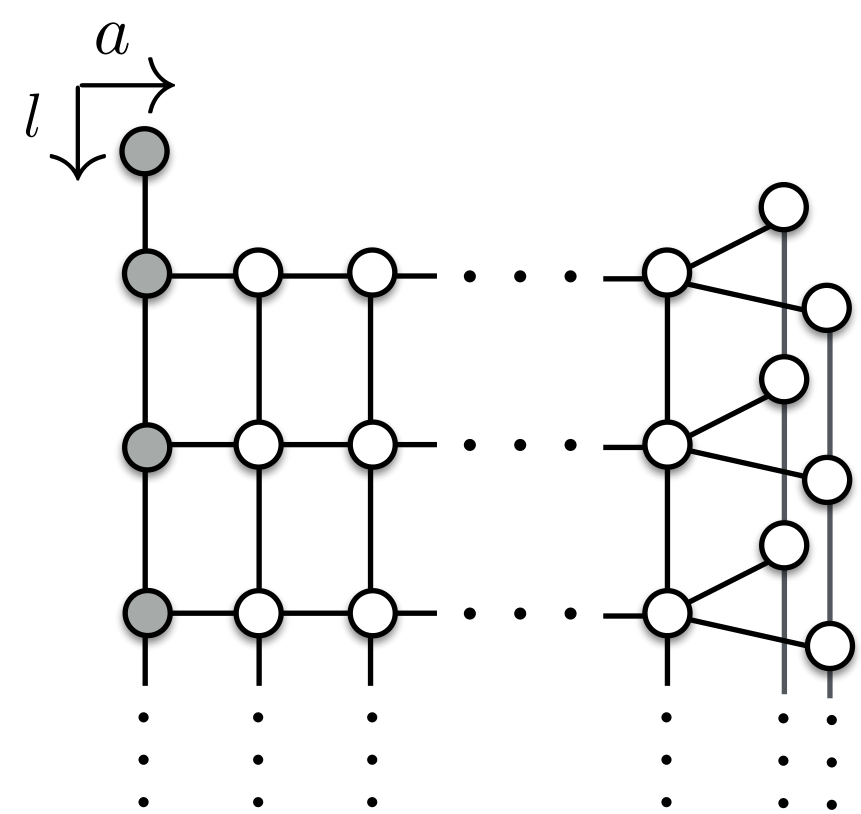
$$\log \mathrm{Y}_{1,l} = C - J\varepsilon_l + \sum_{l'\geq 0} \mathcal{K}_{l,l'} \star \log(1+\mathrm{Y}_{1,l'}) + \sum_{l'\geq 1} \mathrm{K}_{l,l'} \star \log(1+\mathrm{Y}_{2,l'})$$

$$C = J\log\xi^2 - \sum_{l=0}^{\infty} \int [i\partial_u \log f_l(u)]\, \log(1+\mathrm{Y}_{1,l}) \frac{du}{2\pi}$$

- Kernel:

$$\mathcal{K}_{l,l'}(u) = \frac{1}{i}\partial_u \log \mathcal{S}_{l,l'}(u),$$

$$\mathrm{K}_{l,l'}(u) = \sum_{j=(|l-l'|+1)/2}^{(l+l'-1)/2} \frac{2j}{u^2+j^2}$$



TBA for O(D+2) (D even)

❖ **TBA equations:** infinite system of non-linear equations determine Y

- Auxiliary Y-functions: $(a > 1, l \geq 1)$

$$\log \mathrm{Y}_{a,l} = -\sum_{l' \geq 1} \check{\mathrm{K}}_{l,l'} \star \log(1 + \mathrm{Y}_{a,l'}) + \sum_{b, I_{ab} \neq 0} \sum_{l' \geq 1} \mathrm{K}_{l,l'} \star \log(1 + \mathrm{Y}_{b,l'})$$

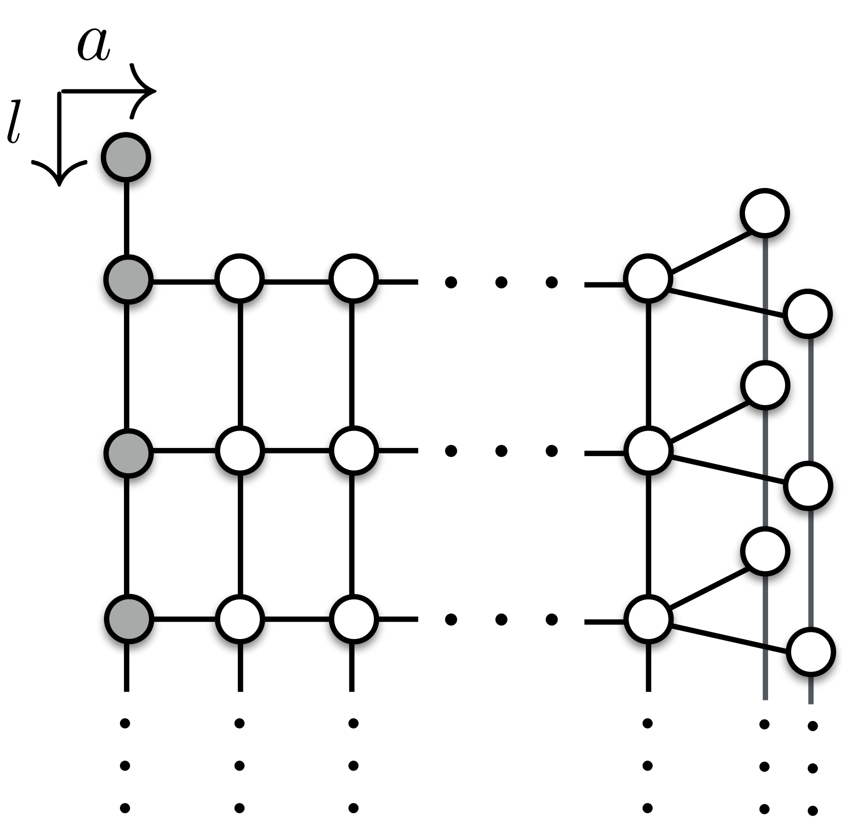
$$\check{\mathrm{K}}_{l,l'} = \mathrm{K}_{l,l'+1} + \mathrm{K}_{l,l'-1}$$

Incidence matrix

- Kernel:

$$\mathcal{K}_{l,l'}(u) = \frac{1}{i} \partial_u \log \mathcal{S}_{l,l'}(u),$$

$$\mathrm{K}_{l,l'}(u) = \sum_{j=(|l-l'|+1)/2}^{(l+l'-1)/2} \frac{2j}{u^2 + j^2}$$



Universal Y-system for O(2r)

❖ **TBA equations can be casted into functional form**

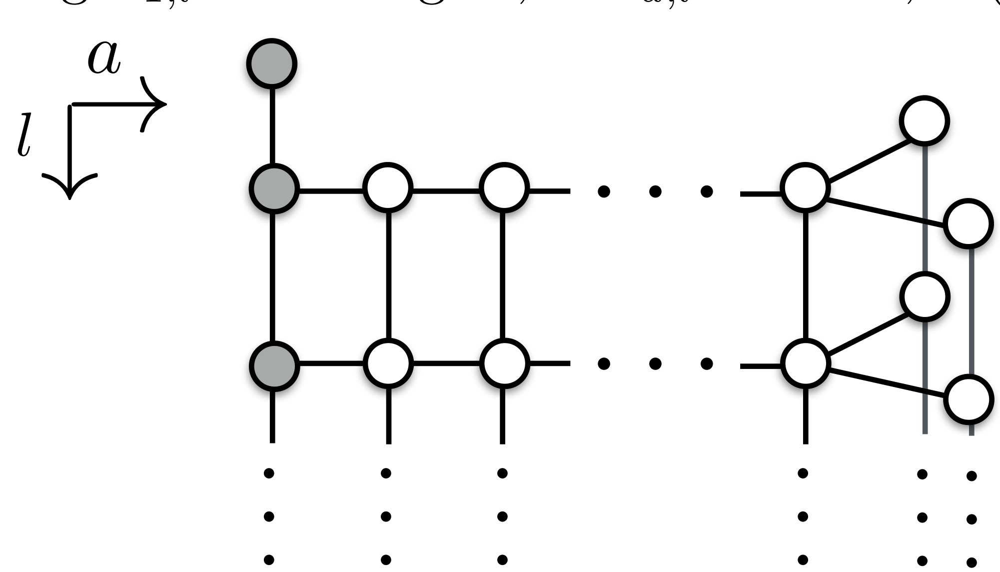
- Universal part: $(1 \le a \le r,\ l \ge 1)$ [Balog, Hegedus '05]

$$\frac{Y^+_{a,l} Y^-_{a,l}}{Y_{a,l+1} Y_{a,l-1}} = \frac{\prod_{b=1}^{r} (1+Y_{b,l})^{I_{ab}}}{(1+Y_{a,l+1})(1+Y_{a,l-1})},$$

- First node: [Balog, Hegedus '05]

$$\frac{1}{Y^{[r-1]}_{1,0} Y^{[1-r]}_{1,0}} = \prod_{k=1}^{r-2} (1+\frac{1}{Y^{[k]}_{r-k-1,1}})(1+\frac{1}{Y^{[-k]}_{r-k-1,1}}) \times (1+\frac{1}{Y_{r-1,1}})(1+\frac{1}{Y_{r,1}})$$

- Asymptotics: $\log Y_{1,l} \sim -\Delta \log u^2, \quad Y_{a,l} \sim \text{const}, \quad (u \to \infty)$



Excited States

❖ **What about $M \neq 0$ spirals?**

- Contour deformation trick: [Dorey, Tateo '96][Bazhanov, Lukyanov, Zamolodchikov '96]

$$\Delta = J\tilde{\delta} - \sum_{l \geq 0} \int_{\mathbb{R}} \frac{du}{2\pi} p_l'(u) \log\left(1 + \mathrm{Y}_{1,l}(u)\right)$$

- Picking zeros of $1 + \mathrm{Y}_{1,l=0}(u_m), \quad (m = 1, \ldots M)$

- TBA equations are the same, just picking some additional driving term!

$$\gamma_M = \Delta - (J\tilde{\delta} + M\delta) = \sum_{m=1}^{M} (2iu_m - \delta) - \sum_{l \geq 0} \int \frac{du}{2\pi} p_l'(u) \log(1 + \mathrm{Y}_{1,l}(u))$$

- How to verify? Go to asymptotic limit!

Asymptotic Bethe Ansatz Equation

❖ **Asymptotic limit:** large J, small coupling

- Exact ABA equation: $1 + \mathrm{Y}_{1,l=0}(u_m) = 0, \quad (m = 1, \dots M)$

$$1 = \xi^{2J} e^{-\varepsilon_0(u_j)J} \prod_{\substack{k=1 \\ k\neq j}}^{M} \mathbb{S}_{0,0}(u_j, u_k), \quad \gamma_M = \sum_{m=1}^{M} (2iu_m - \delta)$$

Scalar

- Single trace operator imposes the cyclicity condition:

$$P_{\mathrm{tot}} = 0 \leftrightarrow 1 = \prod_{j=1}^{M} \xi^2 e^{-\varepsilon_0(u_j)}$$

- Meaning of "asymptotic": valid up to the wrapping order $\mathcal{O}(\xi^{2J})$

- The result matches direct diagrammatic computation up to 3-loops!

$(J = 5, M = 2)$

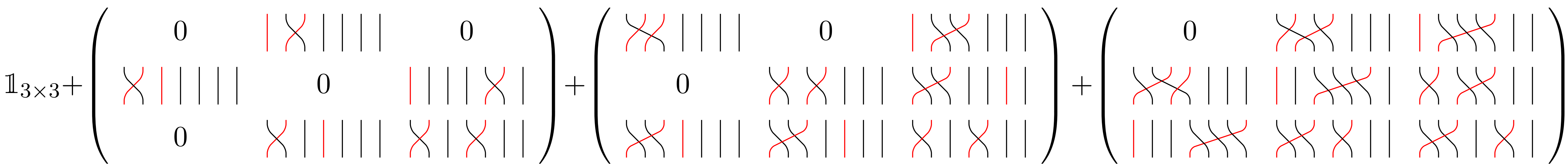
ABA versus Feynman Diagrams

❖ **Diagrams are hard:** operator mixing $(J=5, M=2)$

- Three operators

$$\mathcal{O}_1 = \mathrm{Tr}\left(\phi_2^5\phi_1^2\right), \quad \mathcal{O}_2 = \mathrm{Tr}\left(\phi_2^4\phi_1\phi_2\phi_1\right), \quad \mathcal{O}_3 = \mathrm{Tr}\left(\phi_2^3\phi_1\phi_2^2\phi_1\right).$$

- Two point function: mixing matrix up to $\mathcal{O}(\xi^8)$ 4D case [Caetano, Gurdogan, Kazakov '16]



❖ **ABA is easy:** purely algebraic

- Expanding all Bethe roots: $u_j = -i\delta/2 + \sum_{k=1}^{n_0} u_j^{(k)}\xi^{2k}, \quad (j=1,\cdots,M)$

- Plugging back the solution, we immediately find the eigenvalue!

Summary

❖ **TBA equations for** $\mathcal{O}_{J,M}(x) = \mathtt{Tr}(\phi_1^M \phi_2^J) + \ldots$. **in any D**

- Derive magnon dispersion & S-matrix directly from graphs

- Eigensystem of the graph-building operator $(N = 1, 2)$

- Dual TBA: relates D-dim graph to AdS_{D+1} sigma model

4D, see [Basso, DLZ '19]

Outlooks

❖ **TBA for other types of operators**

- Spinning operator: $D^N \mathcal{O}$
- Conjugate scalar: $\mathtt{Tr}\, \phi_1^M \phi_2^J (\phi_1^\dagger)^{\tilde{M}} (\phi_2^\dagger)^{\tilde{J}}$

❖ **Algebraic reformulation of TBA:** Baxter equation, Quantum Spectrual Curve

E.g. [Ferrando, Frassek, Kazakov '20]

❖ **Computation of higher point functions:** structure constants

E.g. [Basso, Dixson '17] [Grabner, Gromov, Kazakov, Korchemsky '17][Coronado '18] [Gromov, Kazakov, Korchemsky '18] [Derkarchkov, Kazakov, Olivucci '18] [Kazakov, Olivucci, Preti '18] [Korchemsky '19]

❖ **SOV approach to $SO(1, D+1)$ non-compact spin chains** [Fedor's Talk]

[Ferrando] 4D, see also [Derkachov, Olivucci '19]

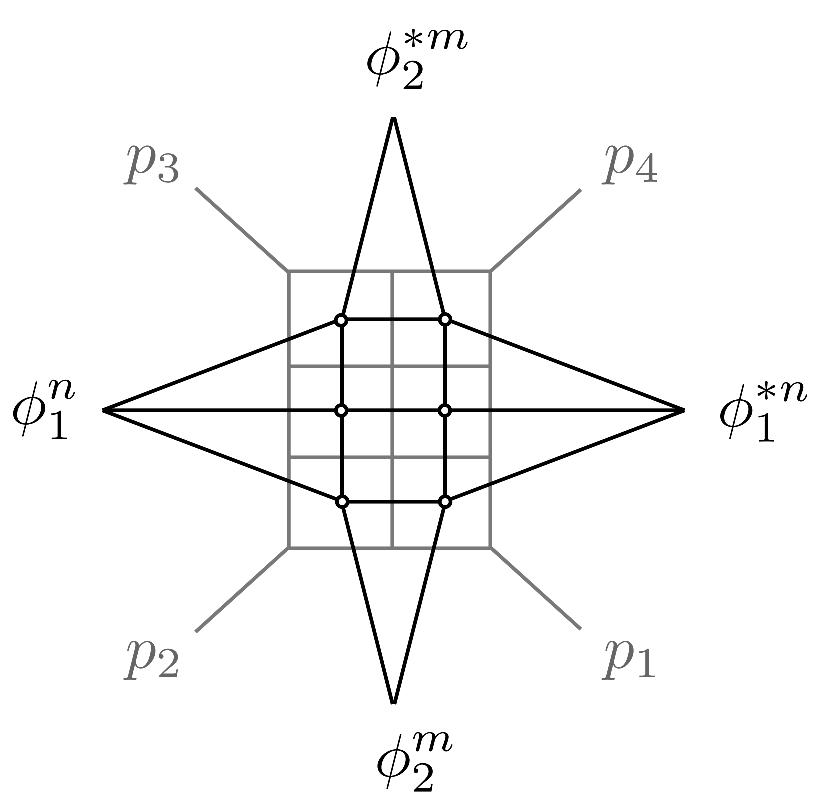
❖ **Holographic dual?** 4D Case: "Fishchain" [Gromov, Sever '19]

Outlooks II

[Basso, Dixson, Kosower, Krajenbrink, DLZ Work in Progress]

❖ **Beyond 2pt functions:** Basso-Dixson 4pt function [Basso, Dixson '17]

Ladder Determinant



BMN Integral

Flux tube Integral

Outlooks II

[Basso, Dixson, Kosower, Krajenbrink, DLZ Work in Progress]

❖ **Beyond 2pt functions:** Basso-Dixson 4pt function

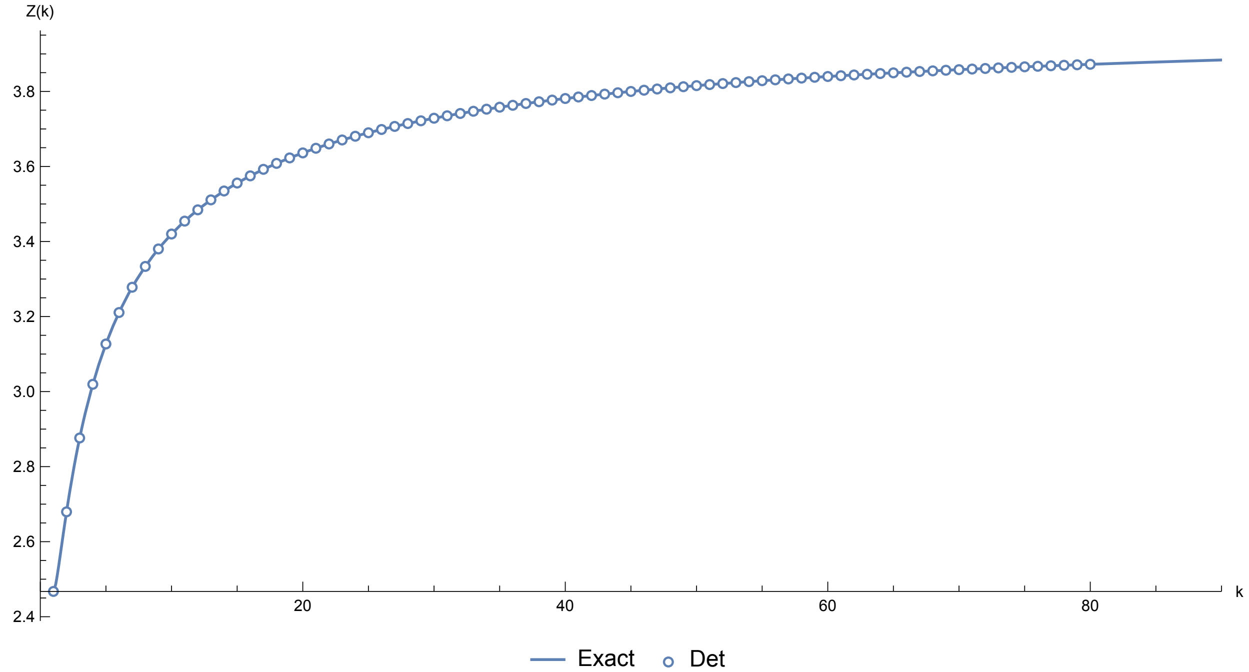
(1) We prove the equivalence of those representations.

(2) We find the exact solution in the thermodynamic limit $m, n \to \infty$, $k = n/m \in [1, \infty)$ fixed

Free energy per site

$$F(k) = \log \pi^2 + k \log \left(\frac{1+\sqrt{1-q}}{2}\right) + \frac{(k-1)^2}{2k} \log K(q) + \frac{1}{k} \log \left(\frac{1-\sqrt{1-q}}{2}\right) - \frac{(k+1)^2}{2k} \log E(q)$$

$$k = \frac{E(q)+\sqrt{1-q}K(q)}{E(q)-\sqrt{1-q}K(q)}, \quad q \in (0,1)$$



Free energy per site depends on k, always greater than Zamolodchikov's result!

Thank you!